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New Stiff Matter Solutions to Einstein Equations

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New exact solutions are presented to the Einstein field equations which are spherically symmetric and static, with a perfect fluid distribution of matter satisfying the equation of state $\rho = p$. One of the obtained solutions may only be used locally, the other represents the stellar interior globally and is singularity-free.

1. INTRODUCTION

With a high degree of precision the interior of a star is a perfect fluid with stress-energy tensor of the form

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + pg^{\mu\nu}$$

where u^{μ} is the velocity 4-vector, and ρ and p are, respectively, the energy density and the pressure of the fluid. Supposing that the spherical star is static, i.e., excluding exploding and pulsating stars, demanding that the star interior (as well as the star exterior) obey Einstein's field equations, we must then resolve these equations with a realistic equation of state in order to have a deeper insight into the stellar interior.

A realistic distribution of matter inside a star must be a polytropic fluid distribution. For such a distribution the Einstein field equations require numerical methods (Tooper, 1964) and one only uses a semi-realistic equation of state, such as $\rho = np$ $(n \geq 1)$. With such a choice Ibanez and Sanz (1982), Klein (1947), Buchdahl and Land (1968), and Whittaker (1968) have already obtained exact closed solutions. In the special case of a stiff matter distribution of fluid, these solutions are not general, as already noted by Ibanez and Sanz.

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The purpose of this paper is to obtain more general solutions with a stiff matter equation of state; two new solutions are then constructed. Study of their physical properties indicates the following.

1. These solutions are physically reasonable, i.e., they present positive pressure and energy density.
2. One of the obtained solutions represents the stellar interior globally, and is singularity-free at the center of the star.
3. The second solution may be used locally, then representing only a partial region of the star.

2. BASIC EQUATIONS

The gravitational field being state and spherically symmetric, isotropic coordinates may be chosen leading to the metric

$$ds^2 = -e^{\nu(r)}\,dt^2 + e^{\lambda(r)}(dr^2 + r^2\,d\Omega^2) \tag{2.1}$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\,d\varphi^2 \tag{2.2}$$

The field (Hajj-Boutros, 1986, 1987) equations read

$$8\pi p = e^{-\lambda}\left[\frac{(\lambda')^2}{4} + \frac{\lambda'\nu'}{2} + \frac{\lambda' + \nu'}{r}\right] \tag{2.3}$$

$$8\pi p = e^{-\lambda}\left[\frac{\lambda''}{2} + \frac{\nu''}{2} + \frac{(\nu')^2}{4} + \frac{\lambda' + \nu'}{2r}\right] \tag{2.4}$$

$$8\pi\rho = e^{-\lambda}\left[\lambda'' + \frac{(\lambda')^2}{4} + \frac{2\lambda'}{r}\right] \tag{2.5}$$

where the prime denotes d/dr, p is the pressure, and ρ is the energy density. The condition of the isotropy of pressure leads to

$$\lambda'' + \nu'' + \tfrac{1}{2}[(\nu')^2 - (\lambda')^2] - \lambda'\nu' - \frac{\lambda' + \nu'}{r} = 0 \tag{2.6}$$

Setting

$$R = \log r \tag{2.7}$$

the field equations become, respectively,

$$8\pi p = \frac{e^{-\lambda}}{r^2}\left(\frac{\dot\lambda^2}{4} + \frac{\dot\lambda\dot\nu}{2} + \dot\lambda + \dot\nu\right) \tag{2.8}$$

$$8\pi p = \frac{e^{-\lambda}}{r^2}\left(\frac{\ddot\lambda + \ddot\nu}{2} + \frac{\dot\nu^2}{4} - \frac{\dot\lambda + \dot\nu}{2}\right) \tag{2.9}$$

$$8\pi\rho = \frac{e^{-\lambda}}{r^2}\left(\ddot\lambda + \frac{\dot\lambda^2}{4} + \dot\lambda\right) \tag{2.10}$$

In turn, the isotropy of pressure condition becomes

$$\ddot{\lambda}+\ddot{\nu}+\tfrac{1}{2}(\dot{\nu}^2-\dot{\lambda}^2)-\dot{\lambda}\dot{\nu}-2(\dot{\nu}+\dot{\lambda})=0 \tag{2.11}$$

where the dot denotes d/dR.

The stiff matter equation of state implies

$$\ddot{\lambda}+\frac{\dot{\lambda}^2}{4}+\dot{\lambda}=\frac{\dot{\lambda}^2}{4}+\frac{\dot{\lambda}\dot{\nu}}{2}+\dot{\lambda}+\dot{\nu} \tag{2.12}$$

By suitable rearrangement we get

$$\frac{\ddot{\lambda}}{\dot{\lambda}/2+1}=\dot{\nu} \tag{2.13}$$

Integrating (2.13), we get

$$\dot{\lambda}=e^{\nu/2}-2 \tag{2.14}$$

Setting:

$$y=e^{\nu/2} \tag{2.15}$$

and introducing relations (2.14) and (2.15) into (2.11), we get

$$4\ddot{y}-2y\dot{y}=y^3-4y \tag{2.16}$$

Setting

$$y=2v$$

we obtain

$$\ddot{v}-v\dot{v}=v(v^2-1) \tag{2.17}$$

which admits the particular solution

$$v=1 \tag{2.18}$$

(the solution $v=-1$ must be rejected since $y=2v=e^{\nu}$). In this case and using formula (2.14) we get

$$\dot{\lambda}=0 \tag{2.19}$$

Thus $\lambda=c$, so we obtain the vacuum solution.

More general solutions to (2.17) are

$$v=\text{th}(\alpha R+\beta) \tag{2.20}$$

$$v=\coth(\alpha R+\beta) \tag{2.21}$$

where α satisfies the equation

$$2\alpha^2+\alpha-1=0 \tag{2.22}$$

β being a constant of integration.

Introducing the values (2.20) and (2.21) into (2.14), we get, respectively,

$$\dot{\lambda}=2[\text{th}(\alpha R+\beta)-1] \tag{2.23}$$

$$\dot{\lambda}=2[\coth(\alpha R+\beta)-1] \tag{2.24}$$

Integrating, we obtain, successively,

$$\lambda=2\{\log[\text{ch}^{1/\alpha}(\alpha R+\beta)]-R\} \tag{2.25}$$

$$\lambda=2\{\log[\text{sh}^{1/\alpha}(\alpha R+\beta)]-R\} \tag{2.26}$$

Taking into account the relation (2.7), we obtain, respectively,

$$e^{\lambda}=\frac{\text{ch}^{2/\alpha}(\alpha\log r+\beta)}{r^2} \tag{2.27}$$

$$e^{\lambda}=\frac{\text{sh}^{2/\alpha}(\alpha\log r+\beta)}{r^2} \tag{2.28}$$

So the line element reads

$$ds^2=-4\,\text{th}^2(\alpha\log r+\beta)\,dt^2+\frac{\text{ch}^{2/\alpha}(\alpha\log r+\beta)}{r^2}(dr^2+r^2\,d\Omega^2) \tag{2.29}$$

[in the case (2.20), (2.27)].

For the case (2.21), (2.28) we get

$$ds^2=-4\coth^2(\alpha\log r+\beta)\,dt^2+\frac{\text{sh}^{2/\alpha}(\alpha\log r+\beta)}{r^2}(dr^2+r^2\,d\Omega) \tag{2.30}$$

Choosing Schwarzschild (or canonical) coordinates obtained making the change $\text{ch}^{1/\alpha}(\alpha\log r+\beta)\to r$ in the case (2.29) of $\text{sh}^{1/\alpha}(\alpha\log r+\beta)\to r$ in the case (2.30), we obtain successively

$$ds^2=-4(1-r^{-2\alpha})\,dt^2+dr^2+r^2\,d\Omega^2 \tag{2.31}$$

$$ds^2=-4(1+r^{-2\alpha})\,dt^2+dr^2+r^2\,d\Omega^2 \tag{2.32}$$

3. PHYSICAL PROPERTIES OF THE SOLUTIONS

Using formula (2.5) and the solutions (2.29) and (2.30), we get, respectively,

$$8\pi p = 8\pi\rho = \frac{1}{\mathrm{ch}^{2/\alpha}(\alpha \log n + \beta)}\left\{\frac{2\alpha}{\mathrm{ch}^2(\alpha \log r + \beta)} + 2[\mathrm{th}(\alpha \log r + \beta) - 1] + [\mathrm{th}(\alpha \log r + \beta) - 1]^2\right\} \tag{3.1}$$

$$8\pi p = 8\pi\rho = \frac{1}{\mathrm{sh}^{2/\alpha}(\alpha \log r + \beta)}\left\{\frac{-2\alpha}{\mathrm{sh}^2(\alpha \log r + \beta)} + 2(\coth(\alpha \log r + \beta) - 1) + [\coth(\alpha \log r + \beta) - 1]^2\right\} \tag{3.2}$$

where α [equation (2.22)] has the values

$$\alpha_1 = +1/2 \tag{3.3}$$

$$\alpha_2 = -1 \tag{3.4}$$

In order to get realistic solutions ($\rho = p > 0$), we must choose respectively α_1 in the case (3.1) and α_2 for (3.2).

At this stage note that for the metric (2.29), r is within the range $r \in [1, \infty[$, since $\mathrm{ch}(\alpha \log r + \beta)$ is always within the same range.

In the case (2.30), r lies in the range $[0, \infty[$.

Consequently, the solution (2.29) can be used locally for the region $r \in [1, \infty[$ and this may serve for studying the local region indicated above. This is not the case of the solution (2.30), which characterizes the whole interior region of a spherical star.

I note here that the expression (3.24) is singularity-free for $\alpha \log r + \beta \to 0$ and the energy density ρ goes to zero since $-2/\alpha = 2$ (α being negative in such a case).

Now in order to compare the present solutions to that of Ibaney and Sanz, I rewrite the pressure and density in canonical coordinates and get

$$8\pi\rho = \frac{1}{r^2}\left[\frac{2\alpha}{r^{2\alpha}} + 2\left(\frac{(r^{2\alpha} - 1)^{1/2}}{r^\alpha} - 1\right) + \left(\frac{(r^{2\alpha} - 1)^{1/2}}{r^\alpha} - 1\right)^2\right] \tag{3.5}$$

in the case (3.1) and

$$8\pi\rho = r^2\left[\frac{-2\alpha}{r^{2\alpha}} + 2\left(\frac{(r^{2\alpha} + 1)^{1/2}}{r^\alpha} - 1\right) + \left(\frac{(r^{2\alpha} + 1)^{1/2}}{r^\alpha} - 1\right)^2\right] \tag{3.6}$$

in the case (3.2).

So the present solutions differ from that of Ibanez and Sanz and consequently from that of Buchdahl and Land.

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